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REVIEW ARTICLE

On the Smarandache Curves of Spatial Quaternionic Involute Curve

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Abstract In this study, the spatial quaternionic curve and the relationship between Frenet frames of involute curve of spatial quaternionic curve are expressed by using the angle between the Darboux vector and binormal vector of the basic curve. Secondly, the Frenet vectors of involute curve are taken as position vector and curvature and torsion of obtained Smarandache curves are calculated. The calculated curvatures and torsions are given depending on Frenet apparatus of basic curve. Finally, an example is given and the shapes of these curves are drawn by using Mapple program.

Keywords Quaternionic curves · Involute curve · Quaternionic Smarandache curves

Mathematics Subject Classification 53A04 · 53C26

1 Introduction

The quaternion first time was introduced by Irish mathematic William Rowan Hamilton in 1843. His initial attempt to generalize the complex numbers by introducing a three-dimensional object failed in the sense that the algebra he constructed for these three-dimensional objects did not have the desired properties. In 1987, Bharathi and Nagaraj defined the quaternionic curves in E^3, E^4 and studied the differential geometry of space curves and introduced Frenet frames and formulae by using quaternions [1]. Following, quaternionic-inclined curves have been defined and harmonic curvatures studied by Karadağ and Sivridağ [2]. In [3], Tuna and Çöken have studied quaternion-valued functions and quaternionic-inclined curves in the semi-Euclidean space ${E_2}^4$. In [4], Erişir and Güngör have obtained some characterizations of semi-real spatial quaternionic rectifying curves in ${IR_1}^3$. Moreover, by the aid of these characterizations, they have investigated semi-real quaternionic rectifying curves in semi-quaternionic space. In [5], after general definition of quaternions, relations between real quaternions and Serret–Frenet formulas have been investigated. Although real quaternions are represented by four basis elements, vectors can also be expressed by using their three basis elements that have complex nature. On the other hand, the difference of quaternion product than the well-known vector product is not an obstacle to obtain Serret–Frenet formulas by real quaternions. In this study, an alternative formulation has been developed for the representation of Serret–Frenet formulas. In the literature, G. Darboux defined the DArboux vector and many studies have been done in the light of this definition. Fenchell gave more importance to Darboux vector interpretation initiated by G. Darboux and he enhanced [6]. The relationship between the Frenet frames of the involute–evolute curve couple for the first time was expressed by using the angle between the Darboux vector and binormal vector of the evolute curve in the Euclidean 3-spaces [7]. Later, Bilici and Çalışkan expressed the transformation matrix between the Frenet frames of the

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involute–evolute curve couple by using the Lorentzian timelike (spacelike) angle between the Darboux vector and binormal vector of the evolute curve according to causal characteristic of the curve couple in the Minkowski 3-spaces [8–10]. In [11] study, matrix representation of the quaternions and general properties of quaternion matrices were given. There is a bijective curve in the set of real quaternions and using the properties of quaternions, the characterizations of involute–evolute curve couples are obtained by Soyfidan [12]. The definitions of spatial quaternionic Smarandache curves according to Bishop frame are given and Frenet and Bishop elements of these curves are calculated. Moreover, in four-dimensional Euclidean space, quaternionic Smarandache curves according to parallel transport frame are defined, Frenet and parallel transport apparatus are calculated by Parlatıcı [13]. Studies about Smarandache curves are available in [14, 15]. The curvature and torsion of the spatial quaternionic Smarandache curve formed by the unit Darboux vector with the normal vector were calculated [16]. The spherical indicatrix curves drawn by quaternionic Frenet vectors were computed. Also the quaternionic geodesic curvatures of the spherical indicatrix curves to $\mathbf{E}^3$ and $\mathbf{S}^2$ were found [17]. In [18], the normal vector and the unit Darboux vector of spatial involute curve of the spatial quaternionic curve are taken as the position vector, the curvature and torsion of obtained Smarandahce curve were calculated.

In this study, an alternative formulation has been developed for the representation of Serret–Frenet formulas. It was observed that a transition could be made between the Serret–Frenet formulas of the main curve and the involute curve.

2 Preliminaries

A real quaternion is defined with q of the form $\mathbf{Q} = \{q | q = d + ae_1 + be_2 + ce_3, d, a, b, c \in \mathbb{R}, e_1, e_2, e_3 \in \mathbb{R}^3\}$ such that

$$\begin{aligned} e_1^2 = e_2^2 = e_3^2 &= -1, \\ e_1 \times e_2 = -e_2 \times e_1 &= e_3, \\ e_1 \times e_3 = -e_3 \times e_1 &= e_2, \\ e_2 \times e_3 = -e_3 \times e_2 &= e_1. \end{aligned} \tag{2.1}$$

We put $S_q = d$ and $V_q = ae_1 + be_2 + ce_3$. Then a quaternion q can be rewritten as $q = S_q + V_q$, where S_q and V_q are the scalar part and vectorial part of q, respectively. Let p and q be any two elements of $\mathbf{Q}$. Then the product of p and q is defined by

$$q_1 \times q_2 = S_{q_1} S_{q_2} - \langle V_{q_1}, V_{q_2} \rangle + S_{q_1} V_{q_2} + S_{q_2} V_{q_1} + V_{q_1} \wedge V_{q_2} \tag{2.2}$$

where we have used the inner product and the cross-product in $\mathbf{R}^3$ [19]. On the other hand, the conjugate of q is denoted by $\bar{q}$ and given by $\bar{q} = S_q - V_q$. These express the symmetric real-valued, non-degenerate, bilinear form as follows,

$$\langle , \rangle|_{\mathbf{Q}} : \mathbf{Q} \times \mathbf{Q} \to \mathbf{R}, \quad \langle q_1, q_2 \rangle|_{\mathbf{Q}} = \frac{1}{2}(q_1 \times \bar{q_2} + q_2 \times \bar{q_1}) \tag{2.3}$$

it is called the quaternionic inner product [19]. Then the norm of q is

$$N(q) = \sqrt{\langle q, q \rangle|_{\mathbf{Q}}} = \sqrt{q \times \bar{q}}, \tag{2.4}$$

A spatial quaternion set defines that $\mathbf{Q}_H = \{q \in \mathbf{Q} | q + \bar{q} = 0\}$ [1]. Let $I = [0, 1]$ be an interval in the real line $\mathbf{R}$ and $s \in I$ be the are-length parameter along the smooth curve [12]

$$\gamma : [0, 1] \to \mathbf{Q}_H, \ \gamma(s) = \sum_{i=1}^{3} \gamma_i(s) e_i, \ \ (1 \le i \le 3). \tag{2.5}$$

The tangent vector $\gamma'(s) = t(s)$ has unit length N(t(s))=1 for alls [1]. Let γ be a differentiable spatial quaternions curve with arc-length parameter s and $\{t(s), n_1(s), n_2(s)\}$ be the Frenet frame of γ at the point $\gamma(s)$,

$$t(s) = \gamma'(s), \qquad n_1(s) = \frac{\gamma''(s)}{N(\gamma''(s))}, \qquad n_2(s) = t(s) \times n_1(s). \tag{2.6}$$

Let $\{t(s), n_1(s), n_2(s)\}$ be the Frenet frame of $\gamma(s)$. Then Frenet formulae, curvature and the torsion are given by

$$\begin{aligned} t'(s) &= k(s) n_1(s), \\ {n_1}'(s) &= -k(s) t(s) + r(s) n_2(s), \\ {n_2}'(s) &= -r(s) n_1(s), \end{aligned} \tag{2.7}$$

where $t(s)$, $n_1(s)$ and $n_2(s)$ are the unit tangent, the unit principal normal and the unit binormal vector of a quaternionic curve, respectively [1, 13]. The functions k, r are called the principal curvature and the torsion, respectively. Let unit speed regular curve $\gamma : [0, 1] \to \mathbf{Q}_H$ and $\gamma^* : [0, 1] \to \mathbf{Q}_H$ be given. For $\forall s \in I$, then the curve γ^* is called the spatial quaternionic involute of the curve γ, if the tangent at the point γ to the curve γ passes through the tangent at the point γ^* to the curve γ^* and $\langle t(s), t^*(s) \rangle|_{\mathbf{Q}} = 0$ [12]. The relations between the Frenet apparatus are as follows [12]

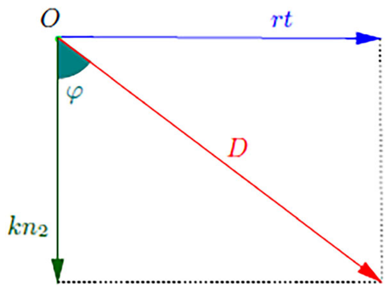

Fig. 1 Darboux vector

$$\begin{aligned} t^*(s) &= n_1(s) \\ {n_1}^*(s) &= \frac{-k(s)}{\sqrt{k(s)^2 + r(s)^2}} t(s) + \frac{r(s)}{\sqrt{k(s)^2 + r(s)^2}} n_2(s) \\ {n_2}^*(s) &= \frac{r(s)}{\sqrt{k(s)^2 + r(s)^2}} t(s) + \frac{k(s)}{\sqrt{k(s)^2 + r(s)^2}} n_2(s) \end{aligned} \tag{2.8}$$

and

$$k^*(s) = \frac{\sqrt{k(s)^2 + r(s)^2}}{| c - s | k(s)}, \quad r^*(s) = \frac{k(s)r'(s) - k'(s)r(s)}{| c - s | k(s)(k^2(s) + r^2(s))}. \tag{2.9}$$

Darboux axis vector of γ spatial quaternions curve indicated by D and this vector is [16] (Fig. 1),

$$D = rt + kn_2. \tag{2.10}$$

If the angle between D and n_2 is φ, it is obtained that

$$\cos\varphi = \frac{k}{\sqrt{k^2 + r^2}}, \qquad \sin\varphi = \frac{r}{\sqrt{k^2 + r^2}}. \tag{2.11}$$

If the unit vector of quaternionic darboux vector indicated by w [16],

$$w = \sin\varphi t + \cos\varphi n_2. \tag{2.12}$$

In [14, 20], Ali and Turgut have introduced some special Smarandache curves in Euclidean and Minkowski space. A regular curve Minkowski space, whose position vector is composed by Frenet vectors on another regular curve, is called Smarandache curve [20]. Special Smarandache curves have been studied by some authors [7, 12, 15, 16, 20, 21]. Let $\gamma = \gamma(s)$ spatial quaternionic curve and $\{t, n_1, n_2\}$ be its moving Frenet–Serret frame. Then we can write Smarandache curve,

$$\beta(s) = \frac{at + bn_1 + cn_2}{\sqrt{a^2 + b^2 + c^2}} \tag{2.13}$$

The relations between the Frenet apparatus are as follows from (2.8), (2.9) and (2.11) then we have [18],

$$\begin{aligned} t^*(s) &= n_1(s) \\ {n_1}^*(s) &= -\cos\varphi t(s) + \sin\varphi n_2(s) \\ {n_2}^*(s) &= \sin\varphi t(s) + \cos\varphi n_2(s), \end{aligned} \tag{2.14}$$

and

$$\begin{aligned} k^*(s) &= \frac{N(D)}{| c - s | k(s)} = \frac{\sec\varphi}{| c - s |} \\ r^*(s) &= \frac{(\tan\varphi)' k(s)}{| c - s | N(D)^2} = \frac{\varphi'}{k(s) | c - s |}. \end{aligned} \tag{2.15}$$

3 Smarandache Curves of Spatial Quaternionic Involute Curve

In this section, we will find Smarandache curves of spatial quaternionic involute curve and equivalent of this Smarandache curves will be written as depending on basic curve, respectively.

Definition 3.1 Let spatial quaternionic γ^* be involute of spatial quaternionic $\gamma(s)$, t^* and ${n_1}^*$ be unit vector of γ^*. In this case, spatial quaternionic Smarandache curve β_1 can be defined by

$$\beta_1(s) = \frac{1}{\sqrt{2}} (t^* + {n_1}^*). \tag{3.1}$$

Theorem 3.1 *Let γ^* be involute of $\gamma(s)$. Frenet apparatus of Smarandache curve β_1 isHerein, coefficients are*

$$
\begin{aligned}
t_{\beta_1(s)} &= \frac{-k^*t^* + k^*n_1^* + r^*n_2^*}{\sqrt{2k^{*2} + r^{*2}}} \\
n_{1\beta_1} &= \frac{\omega_1 t^* + \phi_1 n_1^* + \sigma_1 n_2^*}{\sqrt{\omega_1{}^2 + \phi_1{}^2 + \sigma_1{}^2}} \\
n_{2\beta_1} &= \frac{(k^*\sigma_1 - r^*\phi_1)t^* + (k^*\sigma_1 + r^*\omega_1)n_1^* + (-k^*\phi_1 - k^*\omega_1)n_2^*}{\sqrt{(\omega_1{}^2 + \phi_1{}^2 + \sigma_1^2)(2k^{*2} + r^{*2})}} \\
k_{\beta_1} &= \frac{\sqrt{\omega_1{}^2 + \phi_1{}^2 + \sigma_1{}^2}}{(2k^{*2} + r^{*2})^{\frac{3}{2}}} \\
r_{\beta_1} &= \frac{\sqrt{2}(k^{*2} + r^{*2} - k^{*\prime})(k^*\rho_1 + r^*\eta_1)}{[r^*(2k^{*2} + r^{*2}) + k^*r^{*\prime} - k^{*\prime}r^*]^2 + (k^{*\prime}r^* - k^*r^{*\prime})^2 + (2k^{*3} + k^*r^{*2})^2} \\
&+ \frac{k^*(k^*r^* + r^{*\prime})(\theta_1 + \eta_1)}{[r^*(2k^{*2} + r^{*2}) + k^*r^{*\prime} - k^{*\prime}r^*]^2 + (k^{*\prime}r^* - k^*r^{*\prime})^2 + (2k^{*3} + k^*r^{*2})^2} \\
&+ \frac{(k^{*2} + k^{*\prime})(k^*\rho_1 - r^*\theta_1)}{[r^*(2k^{*2} + r^{*2}) + k^*r^{*\prime} - k^{*\prime}r^*]^2 + (k^{*\prime}r^* - k^*r^{*\prime})^2 + (2k^{*3} + k^*r^{*2})^2}.
\end{aligned}
\tag{3.2}
$$

$$
\begin{aligned}
\omega_1 &= -k^{*2}(2k^{*2} + r^{*2}) - r^*(r^*k^{*\prime} - k^*r^{*\prime}) \\
\phi_1 &= -k^{*2}(2k^{*2} + 3r^{*2}) - r^*(r^{*3} - r^*k^{*\prime} + k^*r^{*\prime}) \\
\sigma_1 &= k^*r^*(2k^{*2} + r^{*2}) - 2k^*(r^*k^{*\prime} - k^*r^{*\prime}) \\
\eta_1 &= k^{*3} + k^*(r^{*2} - 3k^{*\prime}) - k^{*\prime\prime} \\
\theta_1 &= -k^{*3} - k^*(r^{*2} + 3k^{*\prime}) - 3r^*r^{*\prime} + k^{*\prime\prime} \\
\rho_1 &= -k^{*2}r^* - r^{*3} + 2r^*k^{*\prime} + k^*r^{*\prime} + r^{*\prime\prime}.
\end{aligned}
\tag{3.3}
$$

Proof If Smarandache curve β_1's derivative is taken, tangent vector is

$$
t_{\beta_1(s)} = \frac{-k^*t^* + k^*n_1^* + r^*n_2^*}{\sqrt{2k^{*2} + r^{*2}}}. \tag{3.4}
$$

If (3.4) expression is taken derivative, we obtain coefficients where

$$
\begin{aligned}
\omega_1 &= -k^{*2}(2k^{*2} + r^{*2}) - r^*(r^*k^{*\prime} - k^*r^{*\prime}) \\
\phi_1 &= -k^{*2}(2k^{*2} + 3r^{*2}) - r^*(t^{*3} - r^*k^{*\prime} + k^*r^{*\prime}) \\
\sigma_1 &= k^*r^*(2k^{*2} + r^{*2}) - 2k^*(r^*k^{*\prime} - k^*r^{*\prime})
\end{aligned}
\tag{3.5}
$$

impending t'_{β_1} is, we reach

$$
t'_{\beta_1}(s) = \frac{\omega_1 t^* + \phi_1 n_1^* + \sigma_1 n_2^*}{\left(2k^{*2} + r^{*2}\right)^{\frac{3}{2}}}. \tag{3.6}
$$

If the curvature of curve β_1 is shown with curvature k_{β_1}, k_{β_1} is

$$
k_{\beta_1} = \frac{\sqrt{\omega_1{}^2 + \phi_1{}^2 + \sigma_1{}^2}}{(2k^{*2} + r^{*2})^{\frac{3}{2}}}. \tag{3.7}
$$

If principal normal vector of β_1 is shown with $n_{1\beta_1}$, from (2.6) equation $n_{1\beta_1}$ is

$$
n_{1\beta_1} = \frac{\omega_1 t^* + \phi_1 n_1^* + \sigma_1 n_2^*}{\sqrt{\omega_1{}^2 + \phi_1{}^2 + \sigma_1{}^2}}. \tag{3.8}
$$

Because of $n_{2\beta_1} = t_{\beta_1} \times n_{1\beta_1}$, $n_{2\beta_1}$ vector is

$$
n_{2\beta_1} = \frac{(k^*\sigma_1 - r^*\phi_1)t^* + (k^*\sigma_1 + r^*\omega_1)n_1^* + (-k^*\phi_1 - k^*\omega_1)n_2^*}{\sqrt{(\omega_1{}^2 + \phi_1{}^2 + \sigma_1^2)(2k^{*2} + r^{*2})}}. \tag{3.9}
$$

Second and third derivatives of curve β_1 are, respectively,

$$
\beta_1'' = \frac{-(k^{*2} + k^{*\prime})t^* + (k^{*\prime} - k^{*2} - r^{*2})n_1{}^* + (k^*r^* + r^{*\prime})n_2{}^*}{\sqrt{2}},
$$

$$
\beta_1''' = \frac{\eta_1 t^* + \theta_1 n_1{}^* + \rho_1 n_2{}^*}{\sqrt{2}}.
$$

We obtain that coefficients are

$$\begin{aligned} \eta_1 &= k^{*3} + k^*(r^{*2} - 3k^{*\prime}) - k^{*\prime\prime} \\ \theta_1 &= -k^{*3} - k^*(r^{*2} + 3k^{*\prime}) - 3r^*r^{*\prime} + k^{*\prime\prime} \\ \rho_1 &= -k^{*2}r^* - r^{*3} + 2r^*k^{*\prime} + k^*r^{*\prime} + r^{*\prime\prime}. \end{aligned} \tag{3.10}$$

If the torsion of curve β_1 is shown with r_{β_1}, torsion r_{β_1} is

$$\begin{aligned} r_{\beta_1} = &\frac{\sqrt{2}(k^{*2}+r^{*2}-k^{*\prime})(k^*\rho_1+r^*\eta_1)}{[r^*(2k^{*2}+r^{*2})+k^*r^{*\prime}-k^{*\prime}r^*]^2+(k^{*\prime}r^*-k^*r^{*\prime})^2+(2k^{*3}+k^*r^{*2})^2} \\ &+\frac{k^*(k^*r^*+r^{*\prime})(\theta_1+\eta_1)}{[r^*(2k^{*2}+r^{*2})+k^*r^{*\prime}-k^{*\prime}r^*]^2+(k^{*\prime}r^*-k^*r^{*\prime})^2+(2k^{*3}+k^*r^{*2})^2} \\ &+\frac{(k^{*2}+k^{*\prime})(k^*\rho_1-r^*\theta_1)}{[r^*(2k^{*2}+r^{*2})+k^*r^{*\prime}-k^{*\prime}r^*]^2+(k^{*\prime}r^*-k^*r^{*\prime})^2+(2k^{*3}+k^*r^{*2})^2}. \end{aligned} \tag{3.11}$$

□

We can give following corollary

Corollary 3.1 *Frenet apparatus belonging to Smarandache curve β_1 of the quaternionic involute curve is, respectively,*

$$\begin{aligned} t_{\beta_1}(s) &= \frac{(\varphi'\sin\varphi-k)t-N(D)n_1+(\varphi'\cos\varphi+r)n_2}{\sqrt{(\varphi')^2+2N(D)^2}} \\ n_{1\beta_1} &= \frac{\bar{\omega}_1 t+\bar{\phi}_1 n_1+\bar{\sigma}_1 n_2}{\sqrt{\bar{\omega}_1{}^2+\bar{\phi}_1{}^2+\bar{\sigma}_1{}^2}} \\ n_{2\beta_1} &= \frac{-N(D)\bar{\sigma}_1-(\varphi'\cos\varphi+r)\bar{\phi}_1}{\sqrt{\left(\varphi'^2+2N(D)^2\right)(\bar{\omega}_1{}^2+\bar{\phi}_1{}^2+\bar{\sigma}_1{}^2)}}t \\ &+\frac{\bar{\omega}_1(\varphi'\cos\varphi+r)-\bar{\sigma}_1(\varphi'\sin\varphi-k)}{\sqrt{\left(\varphi'^2+2N(D)^2\right)(\bar{\omega}_1{}^2+\bar{\phi}_1{}^2+\bar{\sigma}_1{}^2)}}n_1 \\ &+\frac{\bar{\phi}_1(\varphi'\sin\varphi-k)+\bar{\omega}_1 N(D)}{\sqrt{\left(\varphi'^2+2N(D)^2\right)(\bar{\omega}_1{}^2+\bar{\phi}_1{}^2+\bar{\sigma}_1{}^2)}}n_2 \\ k_{\beta_1} &= \frac{\sqrt{\bar{\omega}_1{}^2+\bar{\phi}_1{}^2+\bar{\sigma}_1{}^2}}{\left(\varphi'^2+2N(D)^2\right)^{\frac{3}{2}}} \\ r_{\beta_1} &= \sqrt{2}\frac{\bar{\eta}_1\widetilde{\omega}_1+\bar{\theta}_1\widetilde{\phi}_1+\bar{\rho}_1\widetilde{\sigma}_1}{\widetilde{\omega}_1^2+\widetilde{\phi}_1^2+\widetilde{\sigma}_1^2} \end{aligned} \tag{3.12}$$

where coefficients are

$$\begin{aligned} \bar{\omega}_1 &= \left(\varphi''\sin\varphi+\varphi'^2\cos\varphi-k'+kN(D)\right)\sqrt{\varphi'^2+2N(D)^2} \\ &\quad-(\varphi'\sin\varphi-k)\left(\sqrt{\varphi'^2+2N(D)^2}\right)' \\ \bar{\phi}_1 &= \left(-N(D)^2-N(D)'\right)\sqrt{\varphi'^2+2N(D)^2} \\ &\quad+N(D)\left(\sqrt{\varphi'^2+2N(D)^2}\right)' \\ \bar{\sigma}_1 &= \left(\varphi''\cos\varphi-\varphi'^2\sin\varphi+r'+rN(D)\right)\sqrt{\varphi'^2+2N(D)^2} \\ &\quad-(\varphi'\cos\varphi+r)\left(\sqrt{\varphi'^2+2N(D)^2}\right)' \\ \bar{\eta}_1 &= \varphi'''\sin\varphi+3\varphi'\varphi''\cos\varphi-\varphi'^3\sin\varphi-k'' \\ &\quad+k'N(D)+2kN(D)'+kN(D)^2 \\ \bar{\theta}_1 &= \varphi'^2N(D)-kk'-rr'+N(D)^3-2N(D)N(D)'+N(D)'' \\ \bar{\rho}_1 &= \varphi'''\cos\varphi-3\varphi'\varphi''\sin\varphi-\varphi'^3\cos\varphi+r''-r'N(D) \\ &\quad-2rN(D)'-rN(D)^2 \\ \widetilde{\omega}_1 &= -N(D)\left(\varphi''\cos\varphi-\varphi'^2\sin\varphi+r'-rN(D)\right) \\ &\quad+\left(N(D)^2+N(D)'\right)(\varphi'\cos\varphi+r) \\ \widetilde{\phi}_1 &= (\varphi'\cos\varphi+r)\left(\varphi''\sin\varphi+\varphi'^2\cos\varphi-k'+kN(D)\right) \\ &\quad-(\varphi'\sin\varphi-k)\left(\varphi''\cos\varphi-\varphi'^2\sin\varphi+r'-rN(D)\right) \\ \widetilde{\sigma}_1 &= (\varphi'\sin\varphi-k)\left(-N(D)^2-N(D)'\right) \\ &\quad+N(D)\left(\varphi''\sin\varphi+\varphi'^2\cos\varphi-k'+kN(D)\right). \end{aligned} \tag{3.13}$$

Proof If (2.14) equation is substituted into (3.1) equation, we obtain that expression depending on basic curve of quaternionic Smarandache curve β_1 is,

$$\beta_1(s) = \frac{1}{\sqrt{2}}(-\cos\varphi(s)t(s)+n_1(s)+\sin\varphi(s)n_2(s)). \tag{3.14}$$

If (2.14) and (2.15) equations are substituted into (3.4) equation, we obtain that tangent vector of quaternionic Smarandache curve β_1 is

$$t_{\beta_1}(s) = \frac{(\varphi'\sin\varphi-k)t-N(D)n_1+(\varphi'\cos\varphi+r)n_2}{\sqrt{(\varphi')^2+2N(D)^2}}. \tag{3.15}$$

From Eqs. (2.15) and (3.5), we can write new coefficients are

$$\begin{aligned}\bar{\omega}_1 &= \left(\varphi'' \sin\varphi + \varphi'^2 \cos\varphi - k' + kN(D)\right)\sqrt{\varphi'^2 + 2N(D)^2}\\ &\quad - (\varphi' \sin\varphi - k)\left(\sqrt{\varphi'^2 + 2N(D)^2}\right)'\\ \bar{\phi}_1 &= \left(-N(D)^2 - N(D)'\right)\sqrt{\varphi'^2 + 2N(D)^2}\\ &\quad + N(D)\left(\sqrt{\varphi'^2 + 2N(D)^2}\right)'\\ \bar{\sigma}_1 &= \left(\varphi'' \cos\varphi - \varphi'^2 \sin\varphi + r' + rN(D)\right)\sqrt{\varphi'^2 + 2N(D)^2}\\ &\quad - (\varphi' \cos\varphi + r)\left(\sqrt{\varphi'^2 + 2N(D)^2}\right)'.\end{aligned} \tag{3.16}$$

If (2.14), (2.15) and (3.16) equations are substituted into derivative expression $t'_{\beta_1}(s)$, the expression in terms of Frenet elements of basic curve of t'_{β_1} derivative vector is found

$$t'_{\beta_1}(s) = \frac{\bar{\omega}_1 t + \bar{\phi}_1 n_1 + \bar{\sigma}_1 n_2}{(\varphi'^2 + 2N(D)^2)^{\frac{3}{2}}}. \tag{3.17}$$

From (2.15), (3.7) and (3.16) equations, the expression depending on curvatures of basic curve of curvature k_{β_1} is

$$k_{\beta_1} = \frac{\sqrt{\bar{\omega}_1{}^2 + \bar{\phi}_1{}^2 + \bar{\sigma}_1{}^2}}{(\varphi'^2 + 2N(D)^2)^{\frac{3}{2}}} \tag{3.18}$$

From (2.14) and (3.8)equations, the expression in terms of Frenet elements of basic curve of vector $n_{1\beta_1}$ is

$$n_{1\beta_1} = \frac{\bar{\omega}_1 t + \bar{\phi}_1 n_1 + \bar{\sigma}_1 n_2}{\sqrt{\bar{\omega}_1{}^2 + \bar{\phi}_1{}^2 + \bar{\sigma}_1{}^2}}. \tag{3.19}$$

If (2.14) and (2.15) equations are substituted into equation, the equivalent in terms of Frenet elements of basic curve of binormal vector $n_{2\beta_1}$ is obtained that

$$\begin{aligned} n_{2\beta_1} &= \frac{-N(D)\bar{\sigma}_1 - (\varphi' \cos\varphi + r)\bar{\phi}_1}{\sqrt{(\varphi'^2 + 2N(D)^2)(\bar{\omega}_1{}^2 + \bar{\phi}_1{}^2 + \bar{\sigma}_1{}^2)}} t\\ &\quad + \frac{\bar{\omega}_1(\varphi' \cos\varphi + r) - \bar{\sigma}_1(\varphi' \sin\varphi - k)}{\sqrt{(\varphi'^2 + 2N(D)^2)(\bar{\omega}_1{}^2 + \bar{\phi}_1{}^2 + \bar{\sigma}_1{}^2)}} n_1\\ &\quad + \frac{\bar{\phi}_1(\varphi' \sin\varphi - k) + \bar{\omega}_1 N(D)}{\sqrt{(\varphi'^2 + 2N(D)^2)(\bar{\omega}_1{}^2 + \bar{\phi}_1{}^2 + \bar{\sigma}_1{}^2)}} n_2. \end{aligned} \tag{3.20}$$

From Eqs. (2.15) and (3.10), new coefficients are as follows,

$$\begin{aligned}\bar{\eta}_1 &= \varphi''' \sin\varphi + 3\varphi'\varphi'' \cos\varphi - \varphi'^3 \sin\varphi - k'' + k'N(D)\\ &\quad + 2kN(D)' + kN(D)^2\\ \bar{\theta}_1 &= \varphi'^2 N(D) - kk' - rr' + N(D)^3 - 2N(D)N(D)' + N(D)''\\ \bar{\rho}_1 &= \varphi''' \cos\varphi - 3\varphi'\varphi'' \sin\varphi - \varphi'^3 \cos\varphi + r''\\ &\quad - r'N(D) - 2rN(D)' - rN(D)^2\\ \widetilde{\omega}_1 &= -N(D)\left(\varphi'' \cos\varphi - \varphi'^2 \sin\varphi + r' - rN(D)\right)\\ &\quad + \left(N(D)^2 + N(D)'\right)(\varphi' \cos\varphi + r)\\ \widetilde{\phi}_1 &= (\varphi' \cos\varphi + r)\left(\varphi'' \sin\varphi + \varphi'^2 \cos\varphi - k' + kN(D)\right)\\ &\quad - (\varphi' \sin\varphi - k)\left(\varphi'' \cos\varphi - \varphi'^2 \sin\varphi + r' - rN(D)\right)\\ \widetilde{\sigma}_1 &= (\varphi' \sin\varphi - k)\left(-N(D)^2 - N(D)'\right)\\ &\quad + N(D)\left(\varphi'' \sin\varphi + \varphi'^2 \cos\varphi - k' + kN(D)\right).\end{aligned} \tag{3.21}$$

The expression in terms of Frenet elements of basic curve of torsion r_{β_1} of Smarandache curve β_1 is as follows,

$$r_{\beta_1} = \sqrt{2}\frac{\bar{\eta}_1 \widetilde{\omega}_1 + \bar{\theta}_1 \widetilde{\phi}_1 + \bar{\rho}_1 \widetilde{\sigma}_1}{\widetilde{\omega}_1^2 + \widetilde{\phi}_1^2 + \widetilde{\sigma}_1^2} \tag{3.22}$$

□

Definition 3.2 Let γ^* be involute of $\gamma(s)$, $n_1{}^*$ and $n_2{}^*$ be unit vector of γ^*. In this case, Smarandache curve β_2 can be defined by

$$\beta_2(s) = \frac{(n_1^* + n_2{}^*)}{\sqrt{2}}. \tag{3.23}$$

Theorem 3.2 Let γ^* be involute of $\gamma(s)$. Frenet apparatus of Smarandache curve β_2 is

$$
\begin{aligned}
t_{\beta_2}(s) &= \frac{-kt^* - rn_1^* + rn_2{}^*}{\sqrt{2r^{*2} + k^{*2}}}\\
n_{1\beta_2} &= \frac{\omega_2 t^* + \phi_2 n_1^* + \sigma_2 n_2^*}{\sqrt{\omega_2{}^2 + \phi_2{}^2 + \sigma_2{}^2}}\\
n_{2\beta_2} &= \frac{-r^*(\sigma_2 + \phi_2)t^* + (r^*\omega_2 + k^*\sigma_2)n_1^* + (-k^*\phi_2 + r^*\omega_1)n_2^*}{\sqrt{(\omega_2{}^2 + \phi_2{}^2 + \sigma_2^2)(2r^{*2} + k^{*2})}}\\
k_{\beta_2} &= \frac{\sqrt{2}\sqrt{\omega_2{}^2 + \phi_2{}^2 + \sigma_2{}^2}}{\left(k^{*2} + 2r^{*2}\right)^2}\\
r_{\beta_2} &= \frac{\sqrt{2}\Big[\big(r^*(2r^{*2} + k^{*2})\big)\eta_2 + (-k^{*\prime} r^* + k^* r^{*\prime})\theta_2 + \big(k^*(k^{*2} + 2r^{*2} + r^{*\prime}) - r^* k^{*\prime}\big)\rho_2\Big]}{\left[r^*(2r^{*2} + k^{*2})\right]^2 + \left[-k^{*\prime} r^* + k^* r^{*\prime}\right]^2 + \left[k^*(k^{*2} + 2r^{*2} + r^{*\prime}) - r^* k^{*\prime}\right]^2}.
\end{aligned}
\tag{3.24}
$$

Herein, coefficients are

$$
\begin{aligned}
\omega_2 &= 2r^{*2}(-k^{*\prime} + r^* r^*) + k^* r^*(k^{*2} + 2r^{*\prime})\\
\phi_2 &= k^*(-k^{*3} - r^{*\prime} k^* + r^* k^{*\prime}) - r^{*2}(3k^{*2} + 2r^{*2})\\
\sigma_2 &= k^{*2}(r^{*\prime} - r^{*2}) - r^*(2r^{*3} + k^* k^{*\prime})\\
\eta_2 &= r^{*3} k^* + k^{*3} + k^{*\prime} r^* + 2k^* r^{*\prime} - k^{*\prime\prime}\\
\theta_2 &= r^{*3} - r^* k^{*2} - 3k^* k^{*\prime} + 3r^{*2} r^{*\prime} - r^{*\prime}\\
\rho_2 &= r^{*3} + r^* k^{*2} - 3r^* r^{*\prime} - r^* r^{*\prime\prime}.
\end{aligned}
\tag{3.25}
$$

Proof The proof is similar to the proof of Theorem 3.2. □

Corollary 3.2 *The Frenet apparatus belonging to Smarandache curve* β_2 *of the quaternionic involute curve is, respectively,*

$$
\begin{aligned}
t_{\beta_2}(s) &= \frac{(\varphi'\cos\varphi + \varphi'\sin\varphi)t - N(D)n_1 + (\varphi'\cos\varphi - \varphi'\sin\varphi)n_2}{\sqrt{2\varphi'^2 + N(D)^2}}\\
n_{1\beta_2} &= \frac{\bar{\omega_2} t + \bar{\phi_2} n_1 + \bar{\sigma_2} n_2}{\sqrt{\bar{\omega_2}{}^2 + \bar{\phi_2}{}^2 + \bar{\sigma_2}{}^2}}\\
n_{2\beta_2} &= \frac{-N(D)\bar{\sigma_2} - (\varphi'\cos\varphi - \varphi'\sin\varphi)\bar{\phi_2}}{\sqrt{(2\varphi'^2 + N(D)^2)(\bar{\omega_2}{}^2 + \bar{\phi_2}{}^2 + \bar{\sigma_2}{}^2)}}t\\
&\quad + \frac{(\varphi'\cos\varphi - \varphi'\sin\varphi)\bar{\omega_2} - (\varphi'\cos\varphi + \varphi'\sin\varphi)\bar{\sigma_2}}{\sqrt{(2\varphi'^2 + N(D)^2)(\bar{\omega_2}{}^2 + \bar{\phi_2}{}^2 + \bar{\sigma_2}{}^2)}}n_1\\
&\quad + \frac{(\varphi'\cos\varphi + \varphi'\sin\varphi)\bar{\phi_2} + N(D)\bar{\omega_2}}{\sqrt{(2\varphi'^2 + N(D)^2)(\bar{\omega_2}{}^2 + \bar{\phi_2}{}^2 + \bar{\sigma_2}{}^2)}}n_2\\
k_{\beta_2} &= \frac{\sqrt{2}\sqrt{\bar{\omega_2}{}^2 + \bar{\phi_2}{}^2 + \bar{\sigma_2}{}^2}}{\left(2\varphi'^2 + N(D)^2\right)^{\frac{3}{2}}}\\
r_{\beta_2} &= \sqrt{2}\,\frac{\bar{\eta_2}\widetilde{\omega}_2 + \bar{\theta_2}\widetilde{\phi}_2 + \bar{\rho_2}\widetilde{\sigma}_2}{\widetilde{\omega}_1^2 + \widetilde{\phi}_2^2 + \widetilde{\sigma}_2^2}
\end{aligned}
\tag{3.26}
$$

where coefficients are

$$
\begin{aligned}
\bar{\omega_2} &= \left(\varphi''\cos\varphi - \varphi'^2\sin\varphi + \varphi''\sin\varphi + \varphi'^2\cos\varphi + kN(D)\right)\\
&\quad \sqrt{2\varphi'^2 + N(D)^2} - (\varphi'\cos\varphi + \varphi'\sin\varphi)\left(\sqrt{2\varphi'^2 + N(D)^2}\right)'\\
\bar{\phi_2} &= \left(N(D)\varphi' - N(D)'\right)\sqrt{2\varphi'^2 + N(D)^2} + N(D)\left(\sqrt{2\varphi'^2 + N(D)^2}\right)'\\
\bar{\sigma_2} &= \left(\varphi''\cos\varphi - \varphi'^2\sin\varphi - \varphi''\sin\varphi - \varphi'^2\cos\varphi - rN(D)\right)\\
&\quad \sqrt{2\varphi'^2 + N(D)^2} - (\varphi'\cos\varphi - \varphi'\sin\varphi)\left(\sqrt{2\varphi'^2 + N(D)^2}\right)'\\
\bar{\eta_2} &= \varphi'''\cos\varphi - 3\varphi'\varphi''\sin\varphi - \varphi'^3\cos\varphi + \varphi'''\sin\varphi\\
&\quad + 3\varphi'\varphi''\cos\varphi - \varphi'^3\sin\varphi + k'N(D) + 2kN(D)' - k\varphi' N(D)\\
\bar{\theta_2} &= 2\varphi'' N(D) + \varphi'^2 N(D) + N(D)^3 + N(D)'\varphi' - N(D)''\\
\bar{\rho_2} &= \varphi'''\cos\varphi - 3\varphi'\varphi''\sin\varphi - \varphi'^3\cos\varphi - \varphi'''\sin\varphi\\
&\quad - 3\varphi'\varphi''\cos\varphi + \varphi'^3\sin\varphi - r'N(D) - 2rN(D)' + rN(D)\varphi'\\
\widetilde{\omega}_2 &= -N(D)\left(\varphi''\cos\varphi - \varphi'^2\sin\varphi + \varphi''\sin\varphi - \varphi'^2\cos\varphi - rN(D)\right)\\
&\quad - (\varphi'\cos\varphi - \varphi'\sin\varphi)\left(N(D)\varphi' - N(D)'\right)
\end{aligned}
$$

$$\widetilde{\phi}_2 = (\varphi' \cos\varphi - \varphi' \sin\varphi) \left(\varphi'' \cos\varphi + \varphi'^2 \sin\varphi + \varphi'' \sin\varphi + \varphi'^2 \cos\varphi + kN(D)\right) - (\varphi' \cos\varphi + \varphi' \sin\varphi) \left(\varphi'' \cos\varphi - \varphi'^2 \sin\varphi - \varphi'' \sin\varphi - \varphi'^2 \cos\varphi - rN(D)\right)$$
$$\widetilde{\sigma}_2 = (\varphi' \cos\varphi + \varphi' \sin\varphi)\left(N(D)\varphi' - N(D)'\right) + N(D)\left(\varphi'' \cos\varphi + \varphi'^2 \sin\varphi + \varphi'' \sin\varphi + \varphi'^2 \cos\varphi + kN(D)\right). \quad (3.27)$$

Proof From (2.14) and (3.23) equations, the expression depending on basic curve of curve $\beta_2(s)$ is

$$\beta_2(s) = \frac{1}{\sqrt{2}}[(\sin\varphi - \cos\varphi)t + (\sin\varphi + \cos\varphi)n_2]. \quad (3.28)$$

Proof is completed from (3.24), (3.25) (2.14) and (2.15) equations. □

Definition 3.3 Let γ^* be involute of $\gamma(s)$, t^* and $n_2{}^*$ be unit vector of γ^*. In this case, Smarandache curve β_3 can be defined by

$$\beta_3(s) = \frac{(t^* + n_2^*)}{\sqrt{2}}. \quad (3.29)$$

Theorem 3.3 Let γ^* be involute of $\gamma(s)$. Frenet apparatus of Smarandache curve β_3 is

$$t_{\beta_3}(s) = \frac{(\varphi' \cos\varphi - k)t + (-\varphi' \sin\varphi + r)n_2}{\sqrt{\varphi'^2 - 2\varphi' N(D) + N(D)^2}}$$
$$n_{1\beta_3} = \frac{\bar{\omega_3} t + \bar{\phi}_3 n_1 + \bar{\sigma_3} n_2}{\bar{\omega_3}^2 + \bar{\phi}_3{}^2 + \bar{\sigma_3}^2}$$
$$n_{2\beta_3} = \frac{\left[(\varphi' \sin\varphi - r)\bar{\phi}_3\right]t + \left[(\varphi' \sin\varphi - r)\bar{\omega_3} - (\varphi' \cos\varphi - k)\bar{\sigma_3}\right]n_1 + \left[(\varphi' \cos\varphi - k)\bar{\phi}_3\right]n_2}{\sqrt{(\bar{\omega_3}^2 + \bar{\phi}_3{}^2 + \bar{\sigma_3}^2)\left(\varphi'^2 - 2\varphi' N(D) + N(D)^2\right)}} \quad (3.32)$$
$$k_{\beta_3} = \frac{\sqrt{2\left(\bar{\omega}_3{}^2 + \bar{\phi}_3{}^2 + \bar{\sigma}_3{}^2\right)}}{\left(\varphi'^2 - 2\varphi' N(D) + N(D)^2\right)^{\frac{3}{2}}}$$
$$r_{\beta_3} = \sqrt{2}\frac{\bar{\eta_3}\widetilde{\omega_3} + \bar{\theta}_3\widetilde{\phi}_3 + \bar{\varphi_3}\widetilde{\sigma_3}}{\widetilde{\omega_3}^2 + \widetilde{\phi}_3{}^2 + \widetilde{\sigma_3}^2}.$$

where coefficients are

$$t_{\beta_3} = n_1{}^*$$
$$n_{1\beta_3} = \frac{-k^* t^* + r^* n_2{}^*}{k^{*2} + r^{*2}}$$
$$n_{2\beta_3} = \frac{r^* t^* + k^* n_2{}^*}{\sqrt{k^{*2} + r^{*2}}}$$
$$k_{\beta_3} = \frac{\sqrt{2(k^{*2} + r^{*2})}}{k^* - r^*}$$
$$r_{\beta_3} = \frac{\sqrt{2}[k^{*3}\rho_3 - 2k^{*2}r^*\rho_3 + k^{*2}r^*\eta_3 + k^* r^{*2}\rho_3 - 2k^* r^{*2}\eta_3 + r^{*3}\eta_3]}{[r^*(k^* - r^*)^2]^2 + [k^*(k^* - r^*)^2]^2}. \quad (3.30)$$

Herein, coefficients are

$$\eta_3 = -3k^* k^{*\prime} + 2k^* r^{*\prime} + k^{*\prime} r^*$$
$$\theta_3 = -k^{*3} + r^* k^{*2} - k^* r^{*2} + r^{*3} + k^{*\prime\prime} - r^{*\prime\prime} \quad (3.31)$$
$$\varphi_3 = k^* r^{*\prime} + 2k^{*\prime} r^* - 3r^* r^{*\prime}.$$

Proof The proof is similar to the proof of Theorem 3.2. □

Corollary 3.3 *The Frenet apparatus belonging to Smarandache curve β_3 of the quaternionic involute curve is, respectively,*

$$
\begin{aligned}
\bar{\omega_3} &= \left(\varphi''' \cos\varphi - \varphi'^2 \sin\varphi - k'\right)\sqrt{\varphi'^2 - 2\varphi' N(D) + N(D)^2} \\
&\quad - (\varphi' \cos\varphi - k)(\varphi'^2 - 2\varphi' N(D) + N(D)^2)' \\
\bar{\phi_3} &= \left(\varphi'^2 N(D) - N(D)^2\right)\sqrt{\varphi'^2 - 2\varphi' N(D) + N(D)^2} \\
\bar{\sigma_3} &= \left(\varphi'' \sin\varphi - \varphi'^2 \cos\varphi + r'\right)\sqrt{\varphi'^2 - 2\varphi' N(D) + N(D)^2} \\
&\quad - (-\varphi' \sin\varphi + r)(\varphi'^2 - 2\varphi' N(D) + N(D)^2)' \\
\bar{\eta_3} &= \varphi''' \cos\varphi - 3\varphi'\varphi'' \sin\varphi - \varphi'^3 \cos\varphi - k'' \\
&\quad - k'\varphi' N(D) + kN(D)^2 \\
\bar{\theta_3} &= \varphi'' N(D) - kk' - rr' + \varphi'' N(D) + \varphi' N(D)' - 2N(D)N(D)' \\
\bar{\varphi_3} &= -\varphi''' \sin\varphi - 3\varphi'\varphi'' \cos\varphi + \varphi'^3 \sin\varphi + r'' \\
&\quad + r'\varphi' N(D) - rN(D)^2 \\
\widetilde{\omega_3} &= (\varphi' \sin\varphi - r)(\varphi' N(D) - N(D)^2) \\
\widetilde{\phi_3} &= (\varphi' \sin\varphi - r)(\varphi'' \cos\varphi - \varphi'^2 \sin\varphi - k') \\
&\quad - (\varphi' \cos\varphi - k)(-\varphi'' \sin\varphi - \varphi'^2 \cos\varphi + r') \\
\widetilde{\sigma_3} &= (\varphi' \cos\varphi - k)(\varphi' N(D) - N(D)^2).
\end{aligned}
\tag{3.33}
$$

Proof From (2.14) and (3.29) equations, the expression depending on basic curve of curve $\beta_2(s)$ is

$$
\beta_3(s) = \frac{1}{\sqrt{2}}[(\sin\varphi t + n_1 + \cos\varphi n_2)]. \tag{3.34}
$$

Proof is completed from (2.14), (2.15), (3.30) and (3.31) equations. □

Definition 3.4 Let γ^* be involute of $\gamma(s)$, t^*, $n_1{}^*$ and $n_2{}^*$ be unit vector of γ^*. In this case, Smarandache curve β_4 can be defined by

$$
\beta_4(s) = \frac{1}{\sqrt{3}}(t^* + n_1{}^* + n_2{}^*). \tag{3.35}
$$

Theorem 3.4 Let γ^* be involute of $\gamma(s)$. Frenet apparatus of Smarandache curve β_4 is

$$
\begin{aligned}
t_{\beta_4}(s) &= \frac{k^* t^* + (k^* - r^*) n_1{}^* + r^* n_2{}^*}{\sqrt{2(k^* + r^* - k^* r^*)}} \\
n_{1\beta_4} &= \frac{\omega_4 t^* + \phi_4 n_1{}^* + \sigma_4 n_2{}^*}{\sqrt{\omega_4{}^2 + \phi_4{}^2 + \sigma_4{}^2}} \\
n_{2\beta_4} &= \frac{((k^* - r^*)\sigma_4 - r^*\phi_4)t^* + (r^*\omega_4 + k^*\sigma_4)n_1{}^* - (k^*\phi_4 + (k^* - r^*)\omega_4)n_2{}^*}{\sqrt{(2k^{*2} + 2r^{*2} - 2k^* r^*)(\omega_4{}^2 + \phi_4{}^2 + \sigma_4{}^2)}} \\
k_{\beta_4} &= \frac{\sqrt{3}}{4}\frac{\sqrt{\omega_4^2 + \phi_4^2 + \sigma_4^2}}{(k^{*2} + r^{*2} - k^* r^*)^2} \\
r_{\beta_4} &= \frac{\sqrt{3}(\eta_4\vartheta_4 + \theta_4\chi_4 + \rho_4\xi_4)}{\vartheta_4^2 + \chi_4^2 + \xi_4^2}.
\end{aligned}
\tag{3.36}
$$

Herein, coefficients are

$$
\begin{aligned}
\omega_4 &= k^{*2}(-2k^{*2} - 4r^{*2} + 4r^* k^* - k^{*2} r^{*\prime}) \\
&\quad + k^* r^*(k^{*\prime} + 2r^{*2} + 2r^{*\prime}) - 2k^{*\prime} r^{*2} \\
\phi_4 &= k^{*2}(-2k^{*2} - 4r^{*2} + 2k^* r^* - r^{*\prime}) \\
&\quad + r^{*2}(-2r^{*2} + 2k^* r^* + k^{*\prime}) + k^* r^*(k^{*\prime} - r^{*\prime}) \\
\sigma_4 &= 2k^{*2}(k^* r^* - 2r^{*2} + r^{*\prime}) + r^{*2}(4k^* r^* - 2r^{*2} + k^{*\prime}) \\
&\quad - k^* r^*(r^{*\prime} + 2k^{*\prime}) \\
\eta_4 &= k^{*\prime} r^* - k^{*\prime\prime} - 3k^* k^{*\prime} + 2k^* r^{*\prime} + k^{*3} + k^* r^{*2} \\
\theta_4 &= r^{*3} - k^{*3} - 3(k^* k^{*\prime} + r^* r^{*\prime}) - (-k^{*\prime\prime} + r^{*\prime\prime}) \\
&\quad + k^* r^*(k^* - r^*) \\
\rho_4 &= r^{*\prime\prime} - k^{*2} r^* - 3r^* r^{*\prime} - r^{*3} + 2r^* k^{*\prime} + k^* r^{*\prime} \\
\vartheta_4 &= 2k^* r^*(k^* - r^*) + k^* r^{*\prime} - r^* k^{*\prime} + 2r^{*3} \\
\chi_4 &= k^* r^{*\prime} - r^* k^{*\prime} \\
\xi_4 &= 2k^{*3} + k^* r^{*\prime} + 2k^* r^{*2} - 2k^{*2} r^* - k^{*\prime} r^*.
\end{aligned}
\tag{3.37}
$$

Proof The proof is similar to the proof of Theorem 3.2. □

Corollary 3.4 *Frenet apparatus belonging to Smarandache curve β_4 of the quaternionic involute curve is, respectively,*

$$
\begin{aligned}
t_{\beta_4} &= \frac{1}{\sqrt{2}}\frac{(\varphi' \cos\varphi + \varphi' \sin\varphi - k)t - N(D)n_1 + (\varphi' \cos\varphi - \varphi' \sin\varphi + r)n_2}{\sqrt{\varphi'^2 - \varphi' N(D) + N(D)^2}} \\
n_{1\beta_4} &= \frac{\bar{\omega_4} t + \bar{\phi_4} n_1 + \bar{\sigma_4} n_2}{\sqrt{\bar{\omega_4}^2 + \bar{\phi_4}^2 + \bar{\sigma_4}^2}} \\
n_{2\beta_4} &= \frac{-N(D)\bar{\sigma_4} - (\varphi' \cos\varphi - \varphi' \sin\varphi + r)\bar{\phi_4}}{\sqrt{2(\varphi'^2 - \varphi' N(D) + N(D)^2)(\bar{\omega_4}^2 + \bar{\phi_4}^2 + \bar{\sigma_4}^2)}} t \\
&\quad + \frac{(\varphi' \cos\varphi - \varphi' \sin\varphi + r)\bar{\omega_4} - (\varphi' \cos\varphi + \varphi' \sin\varphi - k)\bar{\sigma_4}}{\sqrt{2(\varphi'^2 - \varphi' N(D) + N(D)^2)(\bar{\omega_4}^2 + \bar{\phi_4}^2 + \bar{\sigma_4}^2)}} n_1 \\
&\quad + \frac{(\varphi' \cos\varphi + \varphi' \sin\varphi - k)\bar{\phi_4} + N(D)\bar{\omega_4}}{\sqrt{2(\varphi'^2 - \varphi' N(D) + N(D)^2)(\bar{\omega_4}^2 + \bar{\phi_4}^2 + \bar{\sigma_4}^2)}} n_2 \\
k_{\beta_4} &= \frac{\sqrt{3}}{2}\frac{\sqrt{\bar{\omega_4}^2 + \bar{\phi_4}^2 + \bar{\sigma_4}^2}}{(\varphi'^2 - \varphi' N(D) + N(D)^2)^{\frac{3}{2}}} \\
r_{\beta_4} &= \sqrt{3}\frac{\bar{\eta_4}\widetilde{\omega_4} + \bar{\theta_4}\widetilde{\phi_4} + \bar{\rho_4}\widetilde{\sigma_4}}{\widetilde{\omega_4}^2 + \widetilde{\phi_4}^2 + \widetilde{\sigma_4}^2}.
\end{aligned}
\tag{3.38}
$$

where coefficients are

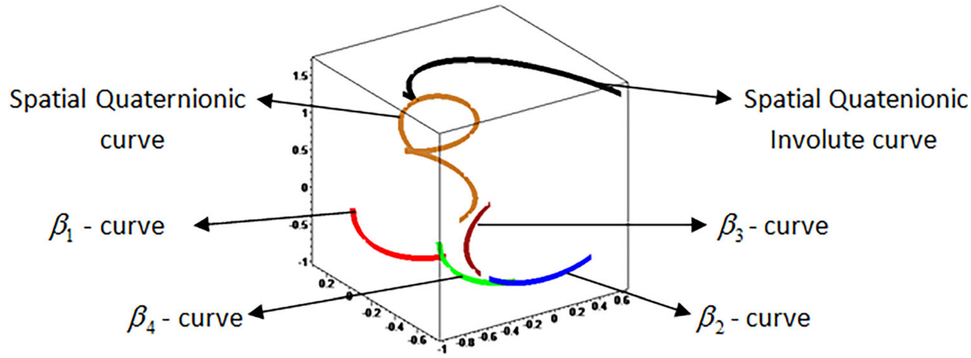

Fig. 2 Spatial quaternionic Smarandache curves

$$\bar{\eta_4} = \varphi''' \cos\varphi - 3\varphi'\varphi'' \sin\varphi - \varphi'^3 \cos\varphi + \varphi''' \sin\varphi$$
$$+ 3\varphi'\varphi'' \cos\varphi - \varphi'^3 \sin\varphi$$
$$- k'' + k'N(D) + 2kN(D)' - kN(D)\varphi' + kN(D)^2$$
$$\bar{\theta_4} = \varphi'' N(D) + \varphi'^2 N(D) + N(D)^3 - kk' - rr'$$
$$+ \varphi' N(D)' + \varphi'' N(D) - 2N(D)N(D)' - N(D)'$$
$$\bar{\rho_4} = \varphi''' \cos\varphi - 3\varphi'\varphi'' \sin\varphi - \varphi'^3 \cos\varphi - \varphi''' \sin\varphi$$
$$- 3\varphi'\varphi'' \cos\varphi + \varphi'^3 \sin\varphi$$
$$+ r'' - r'N(D) - 2rN(D)' + rN(D)\varphi' - rN(D)^2$$
$$\widetilde{\omega}_4 = -N(D)\Big(\varphi'' \cos\varphi - \varphi'^2 \sin\varphi -$$
$$\varphi'' \sin\varphi - \varphi'^2 \cos\varphi + r' - rN(D))$$
$$- \Big(N(D)\varphi' - N(D)^2 - N(D)'\Big)$$
$$(\varphi' \cos\varphi - \varphi' \sin\varphi + r)$$
$$\widetilde{\phi}_4 = (\varphi' \cos\varphi - \varphi' \sin\varphi + r)$$
$$\Big(\varphi'' \cos\varphi - \varphi'^2 \sin\varphi - \varphi'' \sin\varphi + \varphi'^2 \cos\varphi$$
$$- k' + kN(D))$$
$$- (\varphi' \cos\varphi - \varphi' \sin\varphi - k)$$
$$\Big(\varphi'' \cos\varphi - \varphi'^2 \sin\varphi - \varphi'' \sin\varphi - \varphi'^2 \cos\varphi$$
$$+ r' - rN(D))$$
$$\widetilde{\sigma}_4 = (\varphi' \cos\varphi - \varphi' \sin\varphi - k)\Big(N(D)\varphi' - N(D)^2 - N(D)'\Big)$$
$$+ N(D)\Big(\varphi'' \cos\varphi - \varphi'^2 \sin\varphi + \varphi'' \sin\varphi$$
$$+ \varphi'^2 \cos\varphi - k' + kN(D)), \tag{3.39}$$

Proof From (2.14) and (3.35) equations, the expression depending on basic curve of curve $\beta_2(s)$ is

$$\beta_4 = \frac{1}{\sqrt{3}}[(\sin\varphi - \cos\varphi)t + n_1 + (\sin\varphi + \cos\varphi)n_2]. \tag{3.40}$$

Proof is completed from (2.14), (2.15), (3.36) and (3.37) equations. □

Örnek 3.1 *Let be spatial quaternionic curve*

$$\gamma(s) = \left(\frac{\sqrt{2}}{2}\cos\left(\frac{\sqrt{5}}{5}s\right) + \frac{\sqrt{2}}{2}\sin\left(\frac{\sqrt{5}}{5}s\right)\right)e_1$$
$$- \left(\frac{2\sqrt{5}}{5}s\right)e_2 + \left(\frac{\sqrt{2}}{2}\cos\left(\frac{\sqrt{5}}{5}s\right) + \frac{\sqrt{2}}{2}\sin\left(\frac{\sqrt{5}}{5}s\right)\right)e_3$$

Involute curve of curve γ is

$$\gamma^*(s) = \left(\frac{-\sqrt{10}}{10}\sin\left(\frac{\sqrt{5}}{5}s\right) + \frac{\sqrt{10}}{10}s\sin\left(\frac{\sqrt{5}}{5}s\right) + \frac{\sqrt{10}}{10}\cos\left(\frac{\sqrt{5}}{5}s\right)\right.$$
$$\left. - \frac{\sqrt{10}}{10}s\cos\left(\frac{\sqrt{5}}{5}s\right) + \frac{\sqrt{2}}{2}s\cos\left(\frac{\sqrt{5}}{5}s\right) + \frac{\sqrt{2}}{2}s\sin\left(\frac{\sqrt{5}}{5}s\right)\right)e_1$$
$$+ \left(\frac{-2\sqrt{5}}{5}\right)e_2$$
$$+ \left(\frac{\sqrt{10}}{10}\sin\left(\frac{\sqrt{5}}{5}s\right) - \frac{\sqrt{10}}{10}\sin\left(\frac{\sqrt{5}}{5}s\right)s + \frac{\sqrt{10}}{10}\cos\left(\frac{\sqrt{5}}{5}s\right)\right.$$
$$\left. - \frac{\sqrt{10}}{10}s\cos\left(\frac{\sqrt{5}}{5}s\right) - \frac{\sqrt{2}}{2}s\cos\left(\frac{\sqrt{5}}{5}s\right) + \frac{\sqrt{2}}{2}s\sin\left(\frac{\sqrt{5}}{5}s\right)\right)e_3$$

In terms of definition, we obtain special Smarandache curves β_1, β_2, β_3 and β_4 *according to Frenet frame of spatial quaternionic curve* (Fig. 2).

$$t^* = \Big(-\frac{1}{50}\sqrt{10}\cos\big(1/5\sqrt{5}s\big)\sqrt{5} + 1/10\sqrt{10}\sin\big(1/5\sqrt{5}s\big)$$
$$+ \frac{1}{50}\sqrt{10}s\cos\big(1/5\sqrt{5}s\big)\sqrt{5} - \frac{1}{50}\sqrt{10}\sin\big(1/5\sqrt{5}s\big)\sqrt{5}$$
$$- 1/10\sqrt{10}\cos\big(1/5\sqrt{5}s\big) + \frac{1}{50}\sqrt{10}s\sin\big(1/5\sqrt{5}s\big)\sqrt{5}$$
$$+ 1/2\sqrt{2}\cos\big(1/5\sqrt{5}s\big) - 1/10\sqrt{2}s\sin\big(1/5\sqrt{5}s\big)\sqrt{5}$$
$$+ 1/2\sqrt{2}\sin\big(1/5\sqrt{5}s\big) + 1/10\sqrt{2}s\cos\big(1/5\sqrt{5}s\big)\sqrt{5},\ 0,$$
$$\frac{1}{50}\sqrt{10}\cos\big(1/5\sqrt{5}s\big)\sqrt{5} - 1/10\sqrt{10}\sin\big(1/5\sqrt{5}s\big)$$
$$- \frac{1}{50}\sqrt{10}s\cos\big(1/5\sqrt{5}s\big)\sqrt{5} - \frac{1}{50}\sqrt{10}\sin\big(1/5\sqrt{5}s\big)\sqrt{5}$$

$$- 1/10\sqrt{10}\cos\left(1/5\sqrt{5}s\right) + \frac{1}{50}\sqrt{10}s\sin\left(1/5\sqrt{5}s\right)\sqrt{5}$$
$$- 1/2\sqrt{2}\cos\left(1/5\sqrt{5}s\right) + 1/10\sqrt{2}s\sin\left(1/5\sqrt{5}s\right)\sqrt{5}$$
$$+1/2\sqrt{2}\sin\left(1/5\sqrt{5}s\right) + 1/10\sqrt{2}s\cos\left(1/5\sqrt{5}s\right)\sqrt{5}\Big)$$

$$n_1{}^* = \frac{25}{\sqrt{30s^2 - 10s + 505}}\left(\frac{1}{50}\sqrt{10}\sin\left(1/5\sqrt{5}s\right)\right.$$
$$+ 1/25\sqrt{10}\cos\left(1/5\sqrt{5}s\right)\sqrt{5}.$$
$$- \frac{1}{50}\sqrt{10}s\sin\left(1/5\sqrt{5}s\right) - \frac{1}{50}\sqrt{10}\cos\left(1/5\sqrt{5}s\right)$$
$$+ 1/25\sqrt{10}\sin\left(1/5\sqrt{5}s\right)\sqrt{5}$$
$$+ \frac{1}{50}\sqrt{10}s\cos\left(1/5\sqrt{5}s\right) - 1/5\sqrt{2}\sin\left(1/5\sqrt{5}s\right)\sqrt{5}$$
$$- 1/10\sqrt{2}s\cos\left(1/5\sqrt{5}s\right)$$
$$+ 1/5\sqrt{2}\cos\left(1/5\sqrt{5}s\right)\sqrt{5} - 1/10\sqrt{2}s\sin\left(1/5\sqrt{5}s\right), 0,$$
$$- \frac{1}{50}\sqrt{10}\sin\left(1/5\sqrt{5}s\right) - 1/25\sqrt{10}\cos\left(1/5\sqrt{5}s\right)\sqrt{5}$$
$$+ \frac{1}{50}\sqrt{10}s\sin\left(1/5\sqrt{5}s\right)$$
$$- \frac{1}{50}\sqrt{10}\cos\left(1/5\sqrt{5}s\right) + 1/25\sqrt{10}\sin\left(1/5\sqrt{5}s\right)\sqrt{5}$$
$$+ \frac{1}{50}\sqrt{10}s\cos\left(1/5\sqrt{5}s\right)$$
$$+ 1/5\sqrt{2}\sin\left(1/5\sqrt{5}s\right)\sqrt{5} + 1/10\sqrt{2}s\cos\left(1/5\sqrt{5}s\right)$$
$$+ 1/5\sqrt{2}\cos\left(1/5\sqrt{5}s\right)\sqrt{5}$$
$$\left.-1/10\sqrt{2}s\sin\left(1/5\sqrt{5}s\right)\right)$$

$$n_2{}^* = \left(0, -2/5\frac{\sqrt{5}(3s^2 - s + 23)}{\sqrt{30s^2 - 10s + 505}}, 0\right)$$

$$\beta_1 = \left(-\cos\left(\frac{\sqrt{5}}{5}s\right)\right)e_1 - \left(\sin\left(\frac{\sqrt{5}}{5}s\right)\right)e_3,$$

$$\beta_2 = \left(\frac{1}{2}\sin\left(\frac{\sqrt{5}}{5}s\right) - \frac{1}{2}\cos\left(\frac{\sqrt{5}}{5}s\right)\right)e_1 - \left(\frac{\sqrt{2}}{2}\right)e_2$$
$$- \left(\frac{1}{2}\cos\left(\frac{\sqrt{5}}{5}s\right) + \frac{1}{2}\sin\left(\frac{\sqrt{5}}{5}s\right)\right)e_3,$$

$$\beta_3 = \left(-\frac{1}{2}\cos\left(\frac{\sqrt{5}}{5}s\right) - \frac{1}{2}\sin\left(\frac{\sqrt{5}}{5}s\right)\right)e_1 - \left(\frac{\sqrt{2}}{2}\right)e_2$$
$$+ \left(\frac{1}{2}\cos\left(\frac{\sqrt{5}}{5}s\right) - \frac{1}{2}\sin\left(\frac{\sqrt{5}}{5}s\right)\right)e_3,$$

$$\beta_4 = \left(-\frac{\sqrt{6}}{3}\cos\left(\frac{\sqrt{5}}{5}s\right)\right)e_1 - \left(\frac{\sqrt{3}}{3}\right)e_2 + \left(-\frac{\sqrt{6}}{3}\sin\left(\frac{\sqrt{5}}{5}s\right)\right)e_3.$$

4 Conclusion

In this study, we have calculated the Smarandache curves of the involute curve of any curve. To put it simply, we derived curves from a curve according to a method. We found the Frenet frames and curvatures of these curves, which we call Smarandache curves. Finally, we found these results depending on the Frenet frames of the main curve.

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